ASYMMETRIC HYDROMECHANICS

(ASIMMETRICHESKAIA GIDROMEKHANIKA)

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The so-called saymmetric mechanics of continuous media is a generalization of the ordinary continuum mechanics of continuous media for the case in which the stress tensor becomes asymmetric ($\sigma_{ik} \neq \sigma_{ki}$).

The expression $\sigma_{\overline{ik}}\, e_{ikl} = {}^1/_2 (\sigma_{ik} - \sigma_{ki})\, e_{ikl}$ defines the resultant of the moments of the forces which act on an element of the medium. Therefore, this resultant may be balanced only by the moments of couples. In a series of works [1 and 2] the existence of volume-distributed couples m_i, stipulated by external effects, is assumed. However, the case in which the couples act between particles of the medium is of greatest interest [3 to 5], The effect of one contiguous part of the medium on another is then characterized not only by surface forces (stresses), but also by surface moments (micro-moments). This concept is reflected in a series of works [5 to 8].

Initially, asymmetric mechanics of continuous media was developed as a theory of elasticity, but recently investigations have appeared [9 to 11] in which asymmetric hydromechanics is worked out. The need for these investigations stems from the desire to define more precisely the limits of validity of the classical hydromechanics of viscous media. Asymmetric hydromechanics may possibly explain a series of deviations of experimental data from the predictions of theory. It differs from ordinary hydromechanics by a more precise definition of the state of stress which is characterized by an asymmetric stress tensor σ_{ik} ($\sigma_{ik} \neq \sigma_{ki}$) and a tensor of the surface micro-moments μ_{ik}.

The state of deformation is described by the deformation rate tensor $\epsilon_{ik}^{\cdot}$ and by the micro-torsion and micro-bending rate tensor $r_{ik}^{\cdot}$. The rheological laws, i.e. the relation of σ_{ik} and μ_{ik} with $\epsilon_{ik}^{\cdot}$ and $r_{ik}^{\cdot}$, are established. Additional coefficients which may be called the rotational viscosity coefficients appear in these relations.

The equations of motion of the fluid in the velocity components and their general solution are constructed. Boundary conditions sufficient to solve the equations of motion are formulated. Fluid discharge from a capillary, the motion of a sphere in the fluid and the viscosity of suspensions are considered.

1. **The state of stress.** The state of stress of a continuous medium (whether fluid or solid) has been considered from the point of view of asymmertic theory in a series of works [5 to 8]. It is characterized by an asymmetric stress tensor σ_{ik} and a micro-moment tensor μ_{ik} (*), which are sub-

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*) Footnote on the following page.

ject to the equations of the translational and rotational motions of an element of the medium (**)

$$\frac{\partial \sigma_{ik}}{\partial x_k} + \rho f_i = \rho \frac{dv_i}{dt}, \qquad \frac{\partial \mu_{ik}}{\partial x_k} + \sigma_{nm} e_{imn} + \rho m_i = 0 \tag{1.1}$$

Here ρ is the density of the medium, f_i and m_i the density of the volume-distributed forces and moments, v_i the velocity of the translational motion of an element of the medium, d/dt the substantive derivative and ~ the unit axial asymmetric tensor (the Levi-Civita tensor).

It is easily seen that $\sigma_{nm} e_{imn} \equiv \sigma_{nm}^{-} e_{imn}$, where σ_{nm}^{-} is the antisymmetric part of the stress tensor. From (1.1) it follows that the asymmetry of the stress tensor is stipulated by the micro-moments μ_{ik} and the volume-distributed moments m_i. In classical hydromechanics $\mu_{ik} = 0$ and $m_i = 0$; therefore, the stress tensor is symmetric..

2. The dissipation function. To obtain the generalized rheological laws we shall turn to consideration of the process of fluid deformation under isothermal conditions.

In the asymmetric theory it is necessary [5] to take into consideration the "inherent" angular characteristics $\Omega^{\cdot}$ of the particles of the medium which differ from the rotational velocity of a portion of the medium as a whole (i.e. $|\frac{1}{2}\mathrm{rot}+\mathbf{v}|$). In other words, the state of a flowing fluid is determined not only by the field of the translational velocities $\mathbf{v}$, but also by the field of the angular velocities $\Omega^{\cdot}$ (***)

In this case the expression for the work of deformation of a unit volume of the medium (whether fluid or solid) in unit time has the form (****)

$$\partial L \,/\, \partial t = \sigma_{ik} \varepsilon^{\cdot}_{ik} + \mu_{ik} r^{\cdot}_{ik} \tag{2.1}$$

Here and in what follows the dot denotes differentiation with respect to time. The generalized deformation rates $\varepsilon^{\cdot}_{ik}$ and $r^{\cdot}_{ik}$ are related to the velocity field v_i and to the angular velocity field $\Omega^{\cdot}_i$ in the following manner:

$$\varepsilon^{\cdot}_{ik} = \frac{\partial v_i}{\partial x_k} - \Omega^{\cdot}_l e_{lki}, \qquad r^{\cdot}_{ik} = \frac{\partial \Omega^{\cdot}_i}{\partial x_k} \tag{2.2}$$

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*) By micro-moments are meant the density of couples, along with the density of forces (the stresses), which act on a section, conceptually drawn in the medium, which is in a stressed state. The meaning of μ_{ik} is seen from the relation $\mu_{ik} \nu_k = M_i$, where M_i is the density of surface moments on an elementary area inside a body with normal ν_k. The diagonal components of μ_{ik} characterize the torsional moments, the nondiagonal components the bending moments.

**) In the papers [9 to 11] a dynamic term appears in the right-hand side of the second equation of (1.1). However, taking it into account does not affect the basic relations obtained below.

***) This is the conventional terminology. By the angular velocity $\Omega^{\cdot}$ there is meant a quantity averaged over a physically small volume, which characterizes an internal rotational motion in it differing from its motion as a whole.

****) Cf. the expression which follows immediately after relation (8) in [5].

In the case of a fluid the quantity $\partial L/\partial t$ may be represented in the form of a sum of two terms: the rate of increase of free energy $d(\rho F)/dt$ stipulated by the elastic deformations and the rate of heat transfer Ψ stipulated by the enrgy dissipation processes, i.e.

$$\frac{\partial L}{\partial t} = \frac{d}{dt}(\rho F) + \Psi \tag{2.3}$$

If it is assumed that the accumulation of elastic deformation is stipulated only by the bulk compressibility of the fluid, then

$$\frac{d}{dt}(\rho F) = -p\varepsilon^{\cdot}_{ll} \tag{2.4}$$

where p is the thermodynamic pressure. Substituting (2.4) into (2.3) and taking (2.1) into consideration, we obtain

$$\Psi = (\sigma_{ik} + p\delta_{ik})\,\varepsilon^{\cdot}_{ik} + \mu_{ik} r^{\cdot}_{ik} \tag{2.5}$$

Expanding the dissipation function in a series in powers of the components of the deformation rates $\varepsilon^{\cdot}_{ik}$ and $r^{\cdot}_{ik}$ and retaining only terms not higher than second order, we have

$$^1/_2\Psi = A_{iklm}\varepsilon^{\cdot}_{ik}\varepsilon^{\cdot}_{lm} + B_{iklm}\varepsilon^{\cdot}_{ik}r^{\cdot}_{lm} + C_{iklm}r^{\cdot}_{ik}\varepsilon^{\cdot}_{lm} + D_{iklm}r^{\cdot}_{ik}r^{\cdot}_{lm} \tag{2.6}$$

The matrices of the coefficients A_{iklm}, B_{iklm}, C_{iklm} and D_{iklm} are determined by the characteristics of the fluid. We shall restrict our consideration to isotropic fluids only, whose characteristics do not vary with specular reflection (i.e. the fluid is nongyrotropic); then

$$\begin{aligned} ^1/_2\Psi = {} & ^1/_2\lambda\varepsilon^{\cdot}_{nn}\varepsilon^{\cdot}_{kk} + {}^1/_2(\mu+\gamma)\,\varepsilon^{\cdot}_{ik}\varepsilon^{\cdot}_{ki} + {}^1/_2(\mu-\gamma)\,\varepsilon^{\cdot}_{ik}\varepsilon^{\cdot}_{ik} + \\ & + \eta r^{\cdot}_{nn}r^{\cdot}_{kk} + \tau r^{\cdot}_{ik}r^{\cdot}_{ki} + \theta r^{\cdot}_{ik}r^{\cdot}_{ik} \end{aligned} \tag{2.7}$$

The coefficients of this quadratic, essentially positive formula, as shown in [12], obey the following inequalities

$$\mu > 0, \quad 3\lambda + 2\mu > 0, \quad \mu - \gamma > 0, \quad \theta + \tau > 0, \quad \theta - \tau > 0, \quad \gamma < 0$$

3. The rheological laws. To find the rheological laws, we shall apply the theory of Euler concerning homogeneous functions to Expression (2.7)

$$2\Psi = \left(\frac{\partial\Psi}{\partial\varepsilon^{\cdot}_{ik}}\right)_{r^{\cdot}_{ik}}\varepsilon^{\cdot}_{ik} + \left(\frac{\partial\Psi}{\partial r^{\cdot}_{ik}}\right)_{\varepsilon^{\cdot}_{ik}} r^{\cdot}_{ik} \tag{3.1}$$

Identifying this expression with (2.5), we obtain the relations

$$\sigma_{ik} = -p\delta_{ik} + \frac{1}{2}\left(\frac{\partial\Psi}{\partial\varepsilon^{\cdot}_{ik}}\right)_{r^{\cdot}_{ik}}, \qquad \mu_{ik} = {}^1/_2\left(\frac{\partial\Psi}{\partial r^{\cdot}_{ik}}\right)_{\varepsilon^{\cdot}_{ik}} \tag{3.2}$$

Substituting Expression (2.7) for Ψ in them, we obtain

$$\sigma_{ik} = -p\delta_{ik} + \lambda\varepsilon^{\cdot}_{ll}\delta_{ik} + (\mu+\gamma)\,\varepsilon^{\cdot}_{ki} + (\mu-\gamma)\,\varepsilon^{\cdot}_{ik} \tag{3.3}$$

$$\mu_{ik} = 2\eta r^{\cdot}_{nn}\delta_{ik} + 2\tau r^{\cdot}_{ki} + 2\theta r^{\cdot}_{ik} \tag{3.4}$$

These relations, as is easily seen, have the form of a generalized Newton-Navier-Stokes hypothesis. The coefficients λ and μ are the coefficients of the bulk and shear viscosity. The coefficients η , τ and θ may be called the coefficients of rotational viscosity (*).

4. The basic equations. The equations of motion in the velocity components can be obtained by replacing the stresses σ_{ik} and the micro-moments μ_{ik} in the equations of motion in the form (1.1) with the deformation rates $\epsilon^{\cdot}_{ik}$ and $r^{\cdot}_{ik}$ according to the rheological laws (3.3) and (3.4). The latter, in turn, are replaced with the velocities v_i and the angular velocities $\Omega^{\cdot}_i$, in accordance with (2.2).

Neglecting the dependence of the coefficients of viscosity $(\lambda,\mu,\eta,\theta,\gamma)$ on the coordinates, we obtain

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f} - \operatorname{grad} p + (\lambda + 2\mu) \operatorname{grad} \operatorname{div} \mathbf{v} - (\mu - \gamma)] \operatorname{rot} \operatorname{rot} \mathbf{v} - 2\gamma \operatorname{rot} \Omega^{\cdot} \qquad (4.1)$$

$$(\eta + \tau + \theta) \operatorname{grad} \operatorname{div} \Omega^{\cdot} - \theta \operatorname{rot} \operatorname{rot} \Omega^{\cdot} + 2\gamma\Omega^{\cdot} - \gamma \operatorname{rot} \mathbf{v} + \rho \mathbf{m} = 0. \qquad (4.2)$$

This system of two vector equations contains eight unknown functions – three components of velocity $\mathbf{v}$, three components of the "inherent" angular velocity $\Omega^{\cdot}$, the pressure p and the density ρ . To determine these eight functions we must have two additional equations along with the six equations of motion in the form, let us say, of (4.1) and (4.2). They may be obtained from the law of conservation of mass (the continuity equation) and the law of conservation of energy. The continuity equation is not related to the state of stress in the fluid and, therefore, has the same form as in ordinary hydromechanics

$$\partial\rho / \partial t + \operatorname{div} (\rho \mathbf{v}) = 0 \qquad (4.3)$$

The heat transfer equation does depend on the state of stress and, therefore, must be modified. If the concept of specific internal energy ϵ , which is the internal energy unit mass (**), be introduced, its increment will then be determined: (1) by the flow of energy which is stipulated by the mass transfer across the boundaries of the volume, (2) by the transfer of heat due to heat conduction, and (3) by the elementary work of the volume force and the volume moments, the surface forces and the surface moment (***) The first two items do not require modification, but the third, the work of deformation of a unit volume of the fluid, must be calculated according to Formula (2.1). We finally obtain

$$\rho\, d\varepsilon / dt = \operatorname{div} (\varkappa \operatorname{grad} T) + q\, (\sigma_{ik}\varepsilon_{ik}^{\cdot} + \mu_{ik} r^{\cdot}_{ik}) \qquad (4.4)$$

*) They are numerically equal to the moment taken over a unit surface of a portion of the medium, when the latter is rotating with respect to its neighbors so that the gradient of angular velocity is equal to unity. The coefficient γ characterizes the degree of "coupling" of a particle with its environment. If $\gamma \to 0$, then the particle moves about freely relative to the surrounding medium. If $\gamma \to \infty$, then the particle is rotated together with the part of the medium adjacent to it.

**) This can be done neglecting the energy of interaction of the parts of the fluid with each other.

***) Other sources of variation of ϵ will not be considered.

Here $\varkappa$ is the coefficient of heat conduction, q the thermal equivalent of work and T the temperature. To complete the equations we must still add to (4.4) the equation of state

$$p = p(\rho, T) \tag{4.5}$$

5. The general solution of the equations of motion for the case of "creeping" motions. Because of the nonlinearity of the expression for the acceleration

$$d\mathbf{v} / dt = \partial \mathbf{v} / \partial t + (\mathbf{v}\nabla)\,\mathbf{v}$$

the system (4.1) and (4.2) turns out to be very complicated. However, if we restrict our consideration to steady "creeping" motions (the Stokes approximation) in the absence of volume distributed forces (*) and moments ($f_i = 0$, $m_i = 0$), the problem of integrating the equations of motion then reduces to the solution of well-known equations of second order.

In fact, applying the operation of div to (4.1) and (4.2), we obtain

$$\triangle g = 0, \qquad k_1^2 \triangle \varphi - \varphi = 0 \tag{5.1}$$

Here (**)

$$g = (\lambda + 2\mu)\,\mathrm{div}\,\mathbf{v} - p, \qquad \varphi = \mathrm{div}\,\boldsymbol{\Omega}^{\cdot}$$
$$k_1^2 = -\,{}^1\!/_2\,(\eta + \theta + \tau)\,\gamma^{-1} \tag{5.2}$$

Further, substituting (4.2) into (4.1), we arrive at Equation

$$\mathrm{grad}\, g - 2\mu\, \mathrm{rot}\, \boldsymbol{\Omega}^{\cdot} + \theta\,(\mu - \gamma)\,\gamma^{-1}\, \mathrm{rot}\,\mathrm{rot}\,\mathrm{rot}\, \boldsymbol{\Omega}^{\cdot} = 0 \tag{5.3}$$

from which we easily find

$$\mathrm{rot}\, \boldsymbol{\Omega}^{\cdot} = \boldsymbol{\Omega}^{\cdot}_1 + {}^1\!/_2 \mu^{-1}\, \mathrm{grad}\, g \tag{5.4}$$

where $\boldsymbol{\Omega}^{\cdot}_1$ satisfies Equation (***)

$$k_2^2\, \mathrm{rot}\,\mathrm{rot}\, \boldsymbol{\Omega}^{\cdot}_1 + \boldsymbol{\Omega}^{\cdot}_1 = 0, \qquad k_2^2 = {}^1\!/_2 \theta\,(\mu - \gamma)\,\mu^{-1}\gamma^{-1} \tag{5.5}$$

Taking (5.4) into consideration, we obtain from (4.1) a separate equation for

$$-\,\mathrm{grad}\, p + (\lambda + 2\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{v} - \mu\,\mathrm{rot}\,\mathrm{rot}\,\mathbf{v} = 2\gamma\mu\,(\mu - \gamma)^{-1}\,\boldsymbol{\Omega}^{\cdot}_1 \tag{5.6}$$

which may be regarded as inhomogeneous. Its solution has the form

$$\mathbf{v} = \mathbf{v}^{\circ} - \theta\mu^{-1}\boldsymbol{\Omega}^{\cdot}_1 \tag{5.7}$$

Here $\mathbf{v}^{\circ}$ is the solution which corresponds to the homogeneous equation

$$-\,\mathrm{grad}\, p + (\lambda + 2\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{v}^{\circ} - \mu\,\mathrm{rot}\,\mathrm{rot}\,\mathbf{v}^{\circ} = 0 \tag{5.8}$$

i.e. to the Navier-Stokes equations of ordinary hydromechanics, and $-\theta\mu^{-1}\boldsymbol{\Omega}^{\cdot}_1$ is the particular solution of (5.6) which is easily found if (5.8) is considered.

We shall now find $\boldsymbol{\Omega}^{\cdot}$. Substituting the relations (5.5) into (5.4) and taking (5.8) into consideration, we arrive at Equation

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*) Or if the volume forces have a potential.

**) Here $\gamma \neq 0$. The case $\gamma = 0$ requires special consideration.

***) We note that the constants k_1 and k_2 have the dimension of a length.

$$\mathrm{rot}\,\boldsymbol{\Omega}^{\cdot} = -k_2^2\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\Omega}^{\cdot}_1 + {}^1/_2\,\mathrm{rot}\,\mathrm{rot}\,\mathbf{v}^\circ \tag{5.9}$$

from which we find

$$\boldsymbol{\Omega}^{\cdot} = {}^1/_2\,\mathrm{rot}\,\mathbf{v}^\circ - k_2^2\,\mathrm{rot}\,\boldsymbol{\Omega}^{\cdot}_1 + \mathrm{grad}\,\varphi^* \tag{5.10}$$

to within grad φ^* which for the present is undetermined. Comparing the expression for div $\Omega^{\cdot}$ found from (5.10) with Expression (5.2), we obtain $\varphi^* = k_1^2\varphi$ to within an unessential harmonic function. Hence

$$\boldsymbol{\Omega}^{\cdot} = {}^1/_2\,\mathrm{rot}\,\mathbf{v}^\circ - k_2^2\,\mathrm{rot}\,\boldsymbol{\Omega}^{\cdot}_1 + k_1^2\,\mathrm{grad}\,\varphi \tag{5.11}$$

Thus, the general solution of the system of equations for the steady, slow flow of a fluid (4.1) and (4.2) is given by the relations (5.7) and (5.11), i.e. it reduces to the solution of the well-known Navier-Stokes equations of ordinary hydromechanics (5.8) and of equations of the Helmholz type (5.1) and (5.5).

We note that the latter can be written in the form of a single equation, introducing a single vector $\boldsymbol{\Omega}^{\cdot}_2$ in place of φ and $\boldsymbol{\Omega}^{\cdot}_1$ $(\mathrm{div}\,\boldsymbol{\Omega}^{\cdot}_1 = 0)$

$$\boldsymbol{\Omega}^{\cdot}_2 = -k_2^2\,\mathrm{rot}\,\boldsymbol{\Omega}_1^{\cdot} + k_1^2\,\mathrm{grad}\,\varphi \tag{5.12}$$

which satisfies Equation

$$k_1^2\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\Omega}^{\cdot}_2 - k_2^2\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\Omega}^{\cdot}_2 - \boldsymbol{\Omega}^{\cdot}_2 = 0 \tag{5.13}$$

Then in place of (5.7) and (5.11) we have

$$\mathbf{v} = \mathbf{v}^\circ - \theta\mu^{-1}\,\mathrm{rot}\,\boldsymbol{\Omega}^{\cdot}_2, \qquad \boldsymbol{\Omega}^{\cdot} = \boldsymbol{\Omega}^{\cdot}_2 + {}^1/_2\,\mathrm{rot}\,\mathbf{v}^\circ \tag{5.14}$$

6. The boundary conditions. In the asymmetric theory, as has already been noted, the state of the fluid is characterized additionally by a field of angular velocities $\Omega^{\cdot}$. In connection with them three additional equations of motion appear in the theory. It follows from this that the state of the fluid must be characterized by more complicated boundary conditions at the boundary (*).

To solve the equations of motion (4.1) and (4.2) or, what is the same, (5.1), (5.5) and (5.8) six conditions are necessary. The field of translational velocities can obviously be given at the boundary, just as in ordinary hydromechanics, in the form of the no-slip condition

$$[\mathbf{v}^\circ - \theta\mu^{-1}\,\mathrm{rot}\,\boldsymbol{\Omega}^{\cdot}_2]_S = \mathbf{V} \quad (\mathbf{V}\ \text{velocity of the boundary}) \tag{6.1}$$

The second three conditions must obviously determine the field of angular velocities $\Omega^{\cdot}$. In these conditions the mechanism of influence of the boundary on the field of angular velocities $\Omega^{\cdot}$ should have its reflection. Since this mechanism of onteraction is far from clear, we shall then restrict ourselves to the formulation of boundary conditions for the vector $\Omega^{\cdot}$ in certain idealized cases. A limiting case can be a fluid and a solid surface which interact so strongly that a particle of fluid does not turn over rela-

*) Here is meant only the boundary between the fluid and a solid which is an impervious body for the fluid.

tive to the surface and, therefore, its angular velocity is equal to the angular velocity of the surface

$$\Omega^{\cdot}|_S = {}^1/_2 \operatorname{rot} \mathbf{V} \tag{6.2}$$

Another case can also be represented – the limit of weak influence of the surface on the angular velocity of the fluid in which the particles turn over freely with respect to the surface. Then, obviously, the dynamic condition

$$M_i|_S = \mu_{ik}\nu_k|_S = 0 \tag{6.3}$$

must be satisfied on the boundary in place of the kinematic condition (6.2), where $M_i|_S$ is the density of micro-moments on the surface and ν_k is the vector of the normal to it.

Between these extreme cases are found the cases of constrained, rotational slip of the fluid along the surface of the body. In this case it is natural to assume the existence of friction, as a result of which micro-moments arise on the surface. We shall assume that the overfall of the angular velocities $\Delta\Omega^{\cdot}_k = \Omega^{\cdot}_k - {}^1/_2 (\operatorname{rot} \mathbf{V})_k$, on the solid surface is proportional to the surface density of the micro-moments, i.e.

$$\alpha_{ik}\Delta\Omega^{\cdot}_k|_S = M_i|_S \tag{6.4}$$

Here α_{ik} are the coefficients of the rotational surface friction.

Expression (6.4) can be obtained from the rheological law (3.4) (written not in differential form, but in finite differences) by taking the limit $\Delta\xi \to 0$, where $\Delta\xi$ is the thickness of the boundary domain. Moreover, one must consider that only derivatives with respect to the normal to the surface play a fundamental role in (3.4).

If we assume that the fluid and the boundary are isotropic, the boundary domain will then possess cylindrical symmetry with respect to the normal. This reduces the matrix for α_{ik} to the following form:

$$\begin{Vmatrix} 2\alpha & 0 & 0 \\ 0 & 2\alpha & 0 \\ 0 & 0 & \beta \end{Vmatrix} \tag{6.5}$$

in projections on the normal direction (i, k = 1) and on the tangential directions (i, k = 2, 3). On the basis of (6.5) and (3.4) the boundary condition (6.4) shall be written in the form

$$\alpha_{ik}\,(\Omega^{\cdot} - {}^1/_2 \operatorname{rot} \mathbf{V})_k|_S = \left(2\eta \operatorname{div} \Omega^{\cdot}\delta_{ik} + 2\tau \frac{\partial\Omega^{\cdot}_k}{\partial x_i} + 2\theta \frac{\partial\Omega^{\cdot}_i}{\partial x_k}\right)\nu_k\Big|_S \tag{6.6}$$

Any of the three conditions formulated, together with the conditions (6.1), yield six conditions which are sufficient to solve Equations (4.1) and (4.2).

The boundary condition (6.6) contains both of the previously formulated limiting cases. In fact, if $\alpha, \beta \to 0$, the left-hand side of the relation (6.6) vanishes and it turns into the dynamic condition (6.3); if $\alpha, \beta \to \infty$, (6.6) then turns into the condition of the kinematic type (6.2).

How well the formulated boundary conditions reflect the nature of the interaction of the fluid with the solid surface must finally be determined by experiment.

In conclusion we note that in the formulated boundary conditions the vectors $\mathbf{v}^\circ$ and Ω_2° turn out to be "entangled" (confused) on the boundary. The question of their "disentanglement" appears to be mathematically complex and is not solved in the present work.

To illustrate the character of the effects predicted by the asymmetric hydromechanics we shall consider a series of specific problems. For simplicity we shall consider the fluid to be incompressible (div $\mathbf{v} = 0$). Then the integration of the equations of motion reduces to the solution of Equation

$$\operatorname{grad} p = \mu \Delta \mathbf{v}^\cdot \tag{6.7}$$

along with Equations (5.1) and (5,5) with boundary conditions of the form of (6.6).

7. The discharge of fluid from a capillary. We shall consider the discharge of fluid from a cylindrical capillary of circular cross section of radius R. We shall take its axis as the z-axis of a cylindrical system of coordinates (r, φ, z). We shall seek a field of translational velocities $\mathbf{v}$ and of "inherent" angular velocities $\Omega^\cdot$ in the form

$$\mathbf{v} = v(r)\,\mathbf{e}_z, \qquad \Omega^\cdot = \Omega(r)\,\mathbf{e}_\varphi \tag{7.1}$$

For this symmetry of flow $\varphi = 0$, and

$$v(r) = \left[B + \frac{1}{4\mu}\frac{\partial p}{\partial z} r^2 - \frac{\theta}{\mu} C I_0\left(\frac{r}{k_2}\right)\right] \tag{7.2}$$

$$\Omega(r) = \left[C k_2 I_1\left(\frac{r}{k_2}\right) - \frac{1}{4\mu}\frac{\partial p}{\partial z} r\right] \qquad \left(\frac{\partial p}{\partial z} = \text{const}\right) \tag{7.3}$$

Here $I_n(x)$ is the Bessel function of imaginary argument of order n. From the boundary conditions (6.1) and (6.6) we find

$$B = -\frac{1}{4\mu}\frac{\partial p}{\partial z} R^2 + \frac{\theta}{\mu} C I_0(k) \qquad \left(k = \frac{R}{k_2}\right)$$

$$C = \frac{1}{4\mu}\frac{\partial p}{\partial z}\frac{1}{k^{-1} I_1(k) + \delta_1 I_2(k)} \qquad \left(\delta_1 = 1 - \frac{\tau}{\theta} - \frac{\alpha R}{\theta}\right) \tag{7.4}$$

Substituting (7.4) into (7.2) and the latter into (7.1), we find the velocity field

$$\mathbf{v} = -\frac{1}{4\mu}\frac{\partial p}{\partial z} R^2\left[1 - \rho^2 + \frac{2}{A^2}\frac{I_0(k\rho) - I_0(k)}{k^{-1} I_1(k) + \delta_1 I_2(k)}\right]\mathbf{e}_z \qquad \left(\rho = \frac{r}{R},\ A = \frac{\sqrt{2\mu}}{\sqrt{\theta}} R\right) \tag{7.5}$$

The quantity of fluid, issuing from the capillary in unit time, is

$$Q = Q^\circ\left[1 - \frac{4}{A^2}\left(\delta_1 + \frac{I_1(k)}{k I_2(k)}\right)^{-1}\right], \qquad Q^\circ = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial z} \tag{7.6}$$

As seen from (7.6), this quantity is less than that from the Poiseuille formula and also is less the smaller the radius of the cylinder. We shall consider some special cases of Formula (7.6). If $|\gamma|/\mu \gg 1$, then $K \approx A$, and (7.6) takes the form

$$Q = Q^\circ\left[1 - \frac{4}{A}\frac{I_2(A)}{I_1(A)}\right] \qquad (\alpha = \infty) \tag{7.7}$$

$$Q = Q^\circ\left[1 - \frac{4}{A^2}\left(1 + \frac{I_1(A)}{A I_2(A)}\right)^{-1}\right] \qquad (\alpha = 0)$$

The dependence of Q/Q° on A is depicted in Fig.1; curve 1 is for the case of $\alpha = 0$ and curve 2 is for the case $\alpha = \infty$.

If $|\gamma|/\mu$ is comparable to unity, then the curves of Q/Q° against A will depend on $|\gamma|/\mu$ as $A \to 0$ on the ordinate axis. Curve 3 depicts the dependence of Q/Q° on A for $|\gamma|/\mu = 1$ and $\alpha = 0$, and curve 4 for $|\gamma|\mu = 1$ and $\alpha = \infty$.

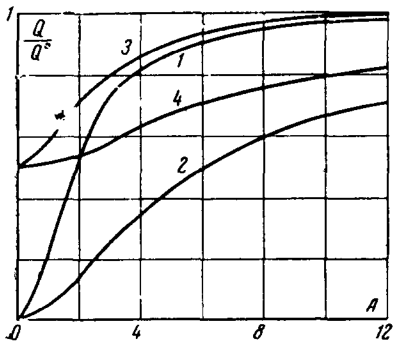

Fig. 1

8. The translational motion of a sphere. We shall consider the problem of flow about a sphere of radius R, located at the origin of the spherical system of coordinates (r, θ, φ), by a stream of viscous fluid which has a specified constant velocity u_i at infinity. The solution of the Navier-Stokes equations, as is well-known[13], has the form

$$v_i^\circ = n_i + e_{ipm}\, e_{pks} \frac{\partial^2}{\partial x_m \partial x_s} [f(r)\, u_k], \qquad f(r) = ar + \frac{b}{r} \tag{8.1}$$

Here and in what follows summation is intended by repeated indices. The solution of Equation (5.5) can be found in the form

$$\Omega^\cdot_{1i} = \ni_{ipm} \ni_{pks} \frac{\partial^2}{\partial x_m \partial x_s} [\psi(r)\, u_k], \qquad \psi(r) = \frac{c}{r} \exp\frac{-r}{k_2} \tag{8.2}$$

Substituting (8.1) and (8.2) into (5.7), we find the field of translational velocities

$$v_i = u_i\,[1 + F_1(r)] + n_i\,(n_k u_k) F_2(r) \tag{8.3}$$

$$F_1(r) = -\left\{\frac{a}{r} + \frac{b}{r^3} - \frac{\theta}{\mu}\left[1 + \frac{r}{k_2} + \frac{r^2}{k_2{}^2}\right]\frac{c}{r^3}\exp\frac{-r}{k_2}\right\}$$

$$F_2(r) = -\left\{\frac{a}{r} - \frac{3b}{r^3} + \frac{\theta}{\mu}\left[3 + 3\frac{r}{k_2} + \frac{r^2}{k_2{}^2}\right]\frac{c}{r^3}\exp\frac{-r}{k_2}\right\} \tag{8.4}$$

Here $n_i = x_i/r$ is the unit radius-vector. We shall rewrite (8.3) in the s spherical system of coordinates

$$v_r = [1 + F_1(r) + F_2(r)]\, u\cos\theta, \qquad v_\theta = -[1 + F_1(r)]\, u \sin\theta, \qquad v_\varphi = 0 \tag{8.5}$$

The field of the "inherent" angular velocities $\Omega^\cdot$, in accordance with (5.11), takes the form

$$\Omega_r = 0, \qquad \Omega_\theta^\cdot = 0, \qquad \Omega_\varphi^\cdot = \Omega(r)\, u \sin\theta$$
$$\Omega(r) = \frac{a}{r^2} - c\left(1 + \frac{r}{k^2}\right)\frac{\exp(-r/k_2)}{r^2} \tag{8.6}$$

The constants a, b and c must be found from the boundary conditions. The boundary condition (6.1) is obviously equivalent to two equalities $1 + F_1(R) = 0$ and $F_2(R) = 0$, from which we find

$$a = \frac{3R}{4}\left\{1 + \frac{4}{3R}c\frac{k^2}{A^2}e^{-k}\right\} \qquad b = \frac{R^3}{4}\left\{1 + \frac{4}{R}c\frac{(k+1)^2+1}{A^2}e^{-k}\right\} \tag{8.7}$$

To determine c there remains the boundary condition (6.6), from which

$$c = {}^3/_4 R[1 - k^2\dot{A}^{-2} + k(1 + \delta_2 k)]^{-1} \qquad \delta_2 = [2 + \tau\theta^{-1} + \alpha R\theta^{-1}]^{-1} \tag{8.8}$$

For the drag F experienced by the sphere we obtain the following expression

$$F = \int_S (\sigma_{rr}\cos\theta - \sigma_{\theta r}\sin\theta)\, ds \tag{8.9}$$

$$\sigma_{rr} = -p + 2\mu\frac{\partial v_r}{\partial r}, \qquad \sigma_{\theta r} = \sigma_{\theta r}^{+} + \sigma_{\theta r}^{-} \tag{8.10}$$

The latter expression contains an antisymmetrical part (with superscript⁻). The pressure p can be found if the explicit expression (8.1) be substituted into (6.7) in place of $\mathbf{v}^\circ$.

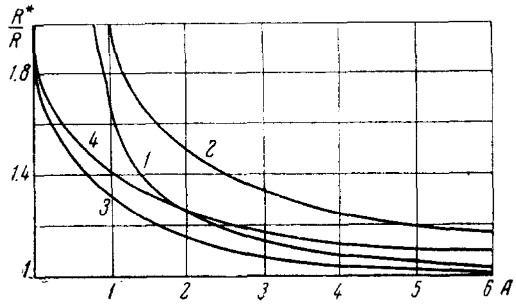

Fig. 2

We obtain

$$p = -2\mu r^{-2} a\,(n_s u_s) \tag{8.11}$$

$$\sigma_{\theta r}^{+}|_{r=R} = \mu\{-aR^{-2} - 3bR^{-4} + \theta\mu^{-1} c\,[3 + 3k + 2k^2 + k^3]\,R^{-4}e^{-k}\}\,u\sin\theta \tag{8.12}$$

$$\sigma_{\theta r}^{-}|_{r=R} = -\theta k_2^{-2} c\,(1+k)\,R^{-2}e^{-k}\,u\sin\theta \tag{8.13}$$

The pressure is reckoned from its value at infinity; therefore, a constant term has been omitted in (8.11). Substituting (8.11) - (8.13) into (8.10) and also taking (8.7) into consideration, we find

$$\sigma_{\theta r}|_{r=R} = -{}^3/_2 R^{-1}\,[1 + {}^4/_3 R^{-1}\, ck^2A^{-2}e^{-k}]\,\mu u\sin\theta \tag{8.14}$$

$$\sigma_{rr}|_{r=R} = {}^3/_2 R^{-1}\,[1 + {}^1/_3 R^{-1}\, ck^2A^{-2}e^{-k}]\,\mu u\cos\theta \tag{8.15}$$

Substituting these expressions into (8.9) and taking (8.8) into consideration, we finally find

$$F = F^\circ\left[1 + \frac{k^2A^{-2}}{1 - k^2A^{-2} + k\,(1+\delta_2 k)}\right], \qquad F^\circ = 6\pi\mu R u \tag{8.16}$$

As is seen from this expression, taking the rotational friction of the fluid particles into account leads to an increase in the resistance in comparison with that given by the Stokes formula. This result is formally equivalent to an increase of the hydrodynamic radius of the sphere. We shall note the limiting cases of Formula (8.16). If $|\gamma|/\mu \gg 1$, then

$$R^* = R\,[1 + A^{-1}] \quad (\alpha = \infty), \qquad R^* = R\,[1 + 2A^{-1}(A+2)^{-1}] \quad (\alpha = 0) \tag{8.17}$$

The dependence of R^*/R on A is depicted in Fig.2 for the case $\alpha = 0$ (curve 1) and $\alpha = \infty$ (curve 2). If $|\gamma|/\mu$ is comparable to unity, the curves R^*/R will tend to a finite value determined by $|\gamma|/\mu$ as $A \to 0$. Curve 3 depicts the dependence of R^*/R on A for the case $|\gamma|/\mu = 1$ and $\alpha = 0$, and curve 4 for the case $|\gamma|/\mu = 1$ and $\alpha = \infty$.

9. The viscosity of suspensions. We shall consider the problem of the viscosity of dilute suspensions with particles of spherical form for the same assumptions which were once made by Einstein [14]. We shall consider then the influence which a bead exerts on the flow of a fluid in which it is immersed. In the asymmetric theory this problem reduces to the solution of Equations (6.7) and (5.5) for the boundary conditions (6.6) and (6.1) on the surface of the bead and the condition

$$v_i\big|_{r\to\infty} = a_{ik}x_k \tag{9.1}$$

at infinity, where a_{ik} is a constant symmetric tensor for which $a_{ii} = 0$. We shall attempt to find a solution to the given problem in the form

$$v_i = a_{ik}x_k + v_{1i}, \qquad \Omega_{1i}^{\cdot} = e_{ipm}\, e_{pks}\, \frac{\partial^2}{\partial x_m \partial x_s}\left[a_{kr}\frac{\partial\psi(r)}{\partial x_r}\right] \tag{9.2}$$

$$v_{1i} = v_{1i}^{\circ} - \frac{\theta}{\mu}\,\Omega^{\cdot}_{1i}, \qquad v_{1i}^{\circ} = e_{ipm}\, e_{pks}\, \frac{\partial^2}{\partial x_m \partial x_s}\left[a_{kr}\frac{\partial f(r)}{\partial x_r}\right] \tag{9.3}$$

Substituting the functions $f(r)$ and $\psi(r)$ in explicit form into (9.2) and (9.3), we find

$$v_{1i}^{\circ} = F_1(r)\,(a_{kr}n_kn_r)\,n_i + F_2(r)\,a_{ir}n_r, \quad \Omega_{1i} = G_1(r)\,(a_{kr}n_kn_r)\,n_i + G_2(r)\,a_{ir}n_r \tag{9.4}$$

$$F_1(r) = 3ar^{-2} - 15br^{-4}, \qquad F_2(r) = 6br^{-4}$$

$$G_1(r) = -c\,(15 + 15rk_2^{-1} + 6r^2k_2^{-2} + r^3k_2^{-3})\,r^{-4}\exp(-r/k_2)$$

$$G_2(r) = -c\,(6 + 6rk_2^{-1} + 3r^2k_2^{-2} + r^3k_2^{-3})\,r^{-4}\exp(-r/k_2) \tag{9.5}$$

Substituting (9.4) and (9.5) into Expression for v_{1i}, and this into (9.2), we find for the field of translational velocities

$$\begin{gathered} v_i = F_1^{\circ}(r)\,(a_{kr}\,n_k\,n_r)n_i + [F_2^{\circ}(r) + r]\,a_{ir}\,n_r \\ F_1^{\circ}(r) = F_1^{\circ}(r) - \theta\mu^{-1}G_1(r), \;\; F_2^{\circ}(r) = F_2(r) - \theta\mu^{-1}G_2(r) \end{gathered} \tag{9.6}$$

In the spherical system of coordinates Expression (9.6) has the form

$$\begin{gathered} v_r = [F_1^{\circ}(r) + F_2^{\circ}(r) + r]\,\Phi_r(\theta,\varphi), \qquad \Phi_r(\theta,\varphi) = a_{kr}\,n_kn_r \\ v_\theta = [F_2^{\circ}(r) + r] - \Phi_\theta(\theta,\varphi), \qquad \Phi_\theta(\theta,\varphi) = a_{ir}\,n_r\,\partial n_i/\partial\theta \\ v_\varphi = [F_2^{\circ}(r) + r]\;\Phi_\varphi(\theta,\varphi), \quad \Phi_\varphi(\theta,\varphi) = a_{ir}\,n_r\csc\theta\,\partial n_i/d\theta \end{gathered} \tag{9.7}$$

Knowing v_{1i}° and $\Omega_{1i}^{\cdot}$, we find from (5.11) the field of "inherent" angular velocities

$$\Omega_r^{\cdot} = 0, \qquad \Omega_\theta^{\cdot} = \Omega(r)\,\Phi_\varphi(\theta,\varphi), \qquad \Omega_\varphi^{\cdot} = -\,\Omega(r)\,\Phi_\theta(\theta,\varphi)$$

$$\Omega(r) = 3ar^{-3} - c\,(3 + 3rk_2^{-1} + r^2k_2^{-2})\,r^{-3}\exp(-r/k_2) \tag{9.8}$$

The constants a, b and c are found from conditions (6.6) and (6.1). As is easily seen from (9.7), condition (6.1) is equivalent to two equalities $(F_1^{\circ}(R)=0,\ F_2^{\circ}(R)+R=0)$, from which

$$a = -\frac{5}{6}R^3\left[1 - \frac{2R}{A^2}\left(G_2(R) + \frac{2}{5}\,G_1(R)\right)\right], \qquad b = -\frac{R^5}{6}\left[1 - \frac{2R}{A^2}G_2(R)\right]$$

The constant c is determined from condition (6.6). We shall write the right-hand side of Expression (6.6) in the symbolic form

$$\mathbf{M} = 2\,(\tau + \theta)\,(\mathbf{e}_r\nabla)\,\boldsymbol{\Omega}^{\cdot} + 2\tau\mathbf{e}_r \times \operatorname{rot}\boldsymbol{\Omega}^{\cdot} \tag{9.9}$$

Taking (9.8) into consideration in it, we obtain

$$\begin{gathered} M_r = 0, \qquad M_\theta = \left[2\theta\frac{d\Omega(r)}{dr} - \frac{2\tau}{r}\,\Omega(r)\right]\Phi_\varphi(\theta,\varphi) \\ M_\varphi = -\left[2\theta\frac{d\Omega(r)}{dr} - \frac{2\tau}{r}\,\Omega(r)\right]\Phi_\theta(\theta,\varphi) \end{gathered} \tag{9.10}$$

From (9.8) and (9.10) it is seen that the boundary condition (6.6) is equivalent to the equality

$$\left[\theta\,\frac{d\Omega(r)}{dr} - (\alpha r + \tau)\,r^{-1}\,\Omega(r)\right]_{r=R} = 0$$

Hence we find (9.11)

$$c = -\frac{5R^3}{2}\frac{e^k}{3(k+1)(1-k^2A^{-2}) + k^2[1+\delta_3(k+1)]}, \quad \delta_3 = \frac{1}{3+\tau\theta^{-1}+\alpha R\theta^{-1}}$$

Substituting the values found for a, b and c into (9.6) and restricting ourselves only to terms of order $1/r^2$, we find

$$v_i = a_{ik}n_k r - \zeta(r)(a_{ps}n_p n_s)n_i, \qquad \zeta(r) = {}^5/_2 R^{*3}r^{-2} \tag{9.12}$$

$$R^{*3} = R^3\left[1 + 3\frac{k^2A^{-2}(k+1)}{3(k+1)(1-k^2A^{-2}) + k^2[1+\delta_3(k+1)]}\right] \tag{9.13}$$

We shall now calculate the dissipation of energy

$$W = \int_s (\sigma_{ik}v_i + \mu_{ik}\Omega_i^{\cdot})n_k\,ds \tag{9.14}$$

The integration is carried out over a sphere of radius $r_0 \gg R$, therefore, only terms of order r^{-2} must be retained in the integrand. From (9.8) it is seen that $\Omega_i^{\cdot} \sim r^{-3}$; consequently $\mu_{ik} \sim r^{-4}$. Therefore, the contribution of the second term in (9.14) to W will be equal to zero. We shall consider the first term. We shall write σ_{ik} in the form $\sigma_{ik} = \sigma_{ik}^+ + \sigma_{ik}^-$, where

$$\sigma_{ik}^+ = -p\delta_{ik} + 2\mu\varepsilon_{ik}^+, \qquad \sigma_{ik}^- = -2\gamma(\varepsilon_{ik}^- + \Omega_l^{\cdot}\,e_{lik}) \tag{9.15}$$

The pressure p can be found if (9.12) is substituted into (6.7)

$$p = -2\mu r^{-1}\zeta(r)\,a_{ps}n_p n_s \tag{9.16}$$

$$\varepsilon_{ik}^+ = a_{ik} + r^{-1}\zeta(r)[5a_{ps}n_p n_s n_i n_k - a_{ks}n_i n_s - a_{is}n_s n_k - \delta_{ik}a_{ps}n_p n_s] \tag{9.17}$$

Substituting (9.16) and (9.17) into (9.15), we find

$$\sigma_{ik}^+ = 2\mu\{a_{ik} + r^{-1}\zeta(r)[5a_{ps}n_p n_s n_i n_k - a_{ks}n_i n_s - a_{is}n_s n_k]\} \tag{9.18}$$

If (9.12) and (9.8) are substituted into (9.15), we then find $\sigma_{\overline{ik}} = 0$. Therefore, in (9.14) only the symmetric part of the stress tensor σ_{ik} makes a contribution. Substituting (9.18) and (9.12) into (9.14) and restricting ourselves only to terms of order r^{-2}, we obtain

$$W = 2\mu\int_s\{3\zeta(r)(a_{ps}n_p n_s)^2 + (r-\zeta(r))\,a_{ps}a_{pk}n_s n_k\}\,ds \tag{9.19}$$

Taking into consideration that

$$\int_s a_{ik}a_{is}n_k n_s\,ds = (a_{11}^2 + a_{22}^2 + a_{33}^2)\,{}^4/_3\,\pi r_0^2$$

$$\int_s (a_{ik}n_i n_k)^2 ds = (a_{11}^2 + a_{22}^2 + a_{33}^2)\,{}^8/_{15}\,\pi r_0^2$$

we find

$$W = 2\mu(a_{11}^2 + a_{22}^2 + a_{33}^2)(V + {}^1/_2\Phi^*) \qquad (V = {}^4/_3\,\pi r_0^3,\ \Phi^* = {}^4/_3\pi R^{*3}) \tag{9.20}$$

Expression (9.20) does not differ from the analogous relation in [14] provided thet the true radius of the bead R be replaced in it by R^*. Omitting further calculation, we find the final result in the form

$$\mu^* = \mu(1 + {}^5/_2\varphi^*), \qquad \varphi^* = {}^4/_3\,\pi R^{*3}c_n \tag{9.21}$$

where μ^* is the viscosity of the suspension and c_n is the volume concentration of beads.

From Formulas (9.13) and (9.21) it is seen that μ^* depends on the radius of the suspended particles in a more complicated manner in the asymmetric theory than in the Einstein formula. If, in accordance with Einstein, μ^* depends only on the total volume occupied by all the beads independent of their radius, in the case under consideration it then depends on the bead radius R.

As was done previously, we shall discuss special cases of Formula (9.13). If $|\gamma|/\mu \gg 1$, then

$$\frac{R^*}{R} = \left[1 + 3\,\frac{(A+1)}{A^2}\right]^{1/2} \quad (\alpha = \infty), \qquad \frac{R^*}{R} = \left[1 + \frac{9\,(A+1)}{A^2\,(A+4)}\right]^{1/2} \quad (\alpha = 0) \tag{9.22}$$

Graphs of the dependencies of (9.22) are analogous to the curves of Fig.2.

The flow of "nonclassical" fluid is accompanied by an additional dispersion sion of energy. In fact, as is seen from the problems under consideration, a solid body moves through the fluid with greater resistence than in the classical case, the quantity of fluid discharged from a capillary turns out to be less than that corresponding in the Poiseuille formula, and the viscosity of suspensions μ^* is greater than that corresponding to the Einstein formula. The additional dispersion of energy also leads to the fact that the velocity field is found to be less in absolute value than in ordinary hydrodynamics for the same flow conditions. We note that, according to (5.7), only the vortical part of the velocity vector experiences a change.

A part of the additional terms in the expression for the dissipation function (2.7) are associated with second derivatives of the velocity field v° with respect to the spatial coordinates, while the classical terms are stipulated only by first derivatives. Such additional characteristics as the tensor of the micro-moments μ_{ik} and the antisymmetric part of the stress tensor are also determined by second derivatives of the field v°. Consequently, the effects associated with the micro-moments are important in inhomogeneous velocity fields.

It is not difficult to see that as $\eta,\ \tau,\ \theta \to 0$ and $\gamma \to \infty$ the results of asymmetric hydromechanics pass over into the results of ordinary hydromechanics. Moreover, as follows from Formulas (5.2) and (5.5), the quantities k_1/l and k_2/l vanish (l is a characteristic dimension of the system). Therefore, the solution of Equations (5.5) and (5.1) has the character of a boundary layer [15]. This means that the additional velocity field is quickly damped in the limiting region of width of the order of k_1 and k_2. Therefore, the nonclassical effects will be the greater, the smaller the linear dimension of the system l (the radius of the capillary, the radius of the bead, etc.). Consequently, it is seen from the problem under consideration (Figs. 1 and 2) that the deviation from the classical results is the greater, the smaller the dimensionless numbers K and A and also that the nonclassical results can be obtained from the formulas of ordinary hydromechanics (for the quantity of flüid discharged, the resistance, the viscosity) by replacing the true dimension of the system l with the effective dimension $l^* = l + \Delta$, where Δ is determined by the characteristics of the fluid.

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